THE QUADRATIC UTILITY MODEL

With the quadratic distribution utility model, legislator i's utility for the Yea outcome on roll call j is just:

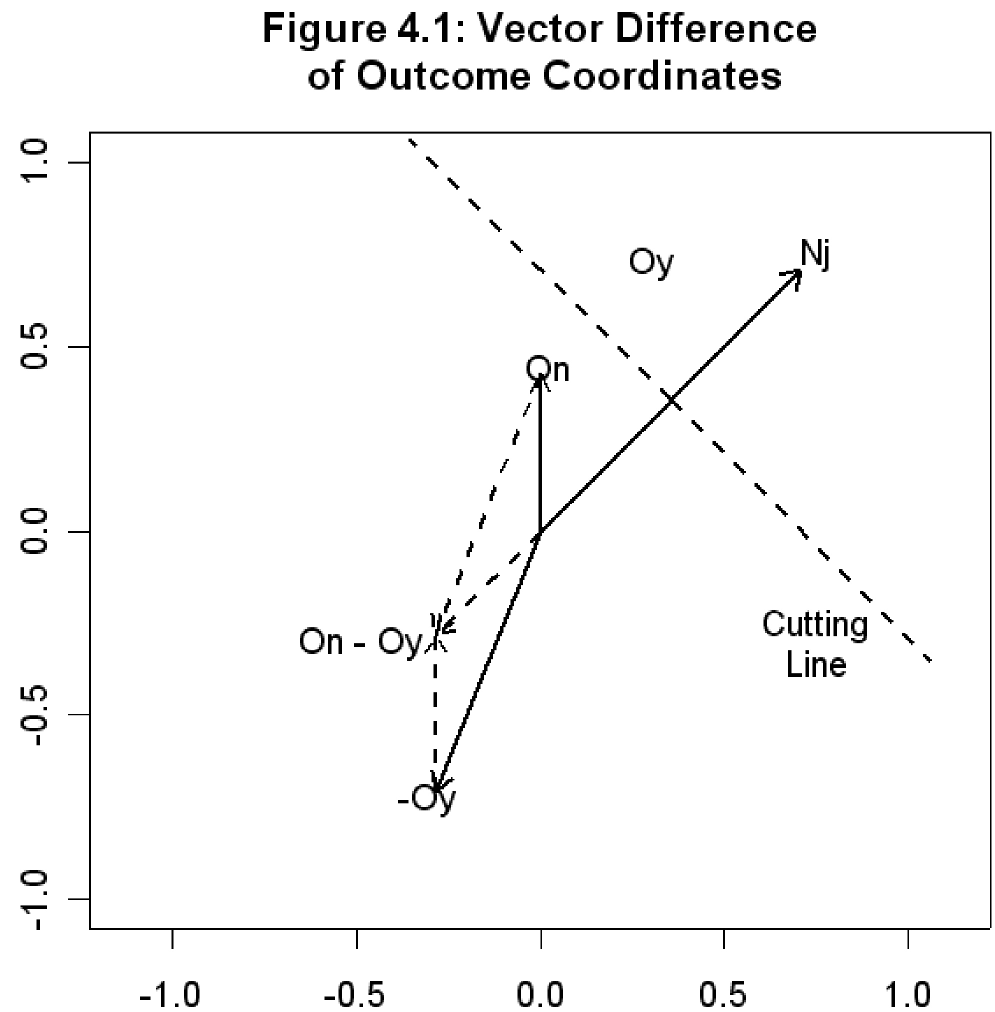
$$u_{ijy} = -d^{2}_{ijy} = -\sum_{k=1}^{s}\left(X_{ik} - O_{jky}\right)^{2} \qquad (4.4)$$

The difference between the two deterministic utility functions is:

$$u_{ijy} - u_{ijn} = -d^{2}_{ijy} + d^{2}_{ijn} = -2\sum_{k=1}^{s} X_{ik}(O_{jkn} - O_{jky}) + \sum_{k=1}^{s}(O_{jkn} - O_{jky})(O_{jkn} + O_{jky}) \qquad (4.5)$$

Now consider the geometry of $\mathbf{O_{jn} + O_{jy}}$ and $\mathbf{O_{jn} - O_{jy}}$ where

$$O_{jn} + O_{jy} = \begin{bmatrix} O_{j1n} + O_{j1y} \\ O_{j2n} + O_{j2y} \\ . \\ . \\ O_{jsn} + O_{jsy} \end{bmatrix} \textbf{ and } O_{jn} - O_{jy} = \begin{bmatrix} O_{j1n} - O_{j1y} \\ O_{j2n} - O_{j2y} \\ . \\ . \\ O_{jsn} - O_{jsy} \end{bmatrix}$$



So that the vector $\mathbf{O_{jn} - O_{jy}}$ is simply a constant times the normal vector:

$$\gamma_j \mathbf{N_j} = \mathbf{O_{jn} - O_{jy}} \tag{4.6}$$

where

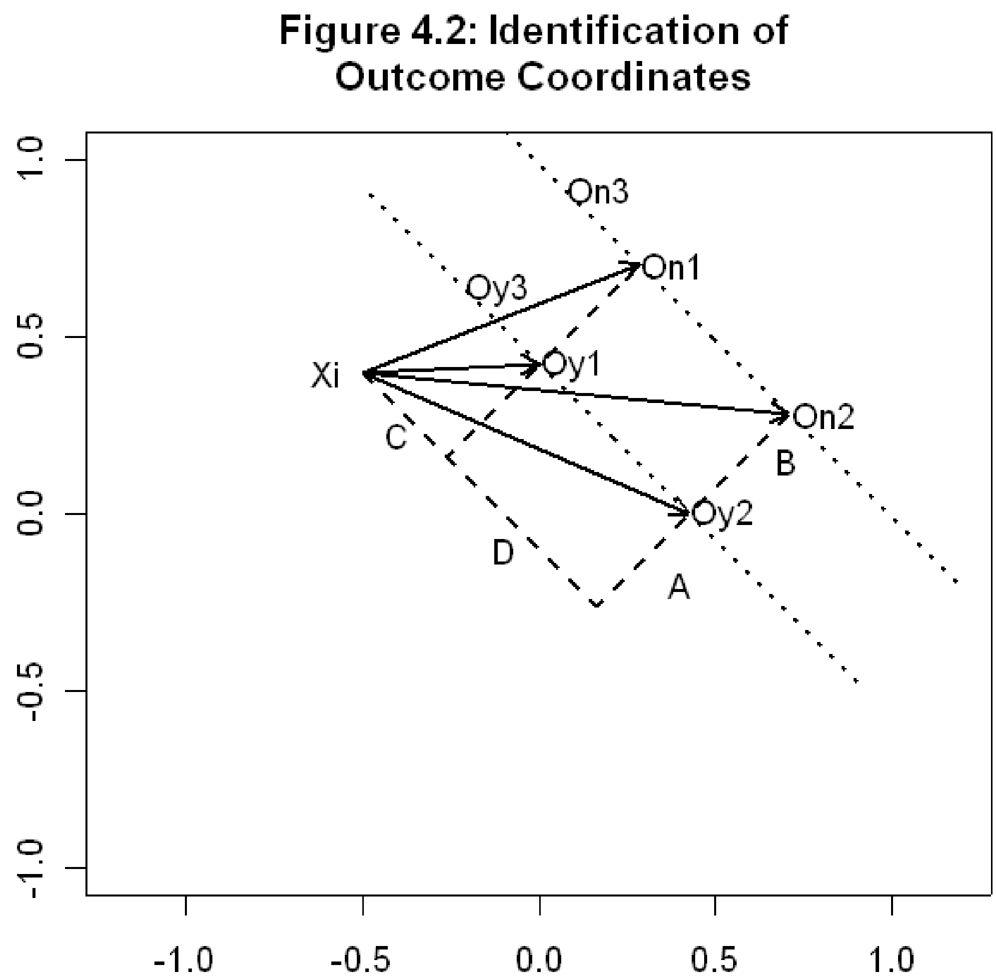
$$\gamma_j = \left[\sum_{k=1}^{s}(O_{jkn} - O_{jky})^2\right]^{\frac{1}{2}} \text{ if } \mathbf{O_{jn}'N_j} > \mathbf{O_{jy}'N_j} \text{ or}$$

$$\gamma_j = -\left[\sum_{k=1}^{s}(O_{jkn} - O_{jky})^2\right]^{\frac{1}{2}} \text{ if } \mathbf{O_{jn}'N_j} < \mathbf{O_{jy}'N_j}$$

and the vector $\mathbf{O_{jn}} + \mathbf{O_{jy}}$ divided by 2 is the midpoint of the Yea and Nay outcomes:

$$\mathbf{Z_j} = \frac{\mathbf{O_{jn}} + \mathbf{O_{jy}}}{2}$$

The outcomes are only identified up to parallel tracks through the space:



$$u_{iy} - u_{in} = -d_{iy}^2 + d_{in}^2 = -(C^2 + A^2) + [C^2 + (A+B)^2] = 2AB + B^2$$

$$u_{iy} - u_{in} = -d_{iy}^2 + d_{in}^2 = -[(C+D)^2 + A^2] + [(C+D)^2 + (A+B)^2] = 2AB + B^2$$

This allows equation (4.5) to be rewritten as the vector equation:

$$u_{ijy} - u_{ijn} = -d_{iy}^2 + d_{in}^2 = 2\gamma_j(\mathbf{Z_j'N_j - X_i'N_j}) = 2\gamma_j(c_j - w_i) \qquad (4.7)$$

If you assume that the error process is normally distributed:

$$\varepsilon_{ijn} - \varepsilon_{ijy} \sim N(0, 1)$$

Then the distribution of the difference between the overall utilities is

$$U_{ijy} - U_{ijn} \sim N(u_{ijy} - u_{ijn}, 1)$$

Hence the probability that legislator i votes Yea on the j[th] roll call can be rewritten as:

$$\mathbf{P_{ijy} = P(}U_{ijy} > U_{ijn}\mathbf{) = P(}\varepsilon_{ijn} - \varepsilon_{ijy} < u_{ijy} - u_{ijn}\mathbf{) = \Phi[}u_{ijy} - u_{ijn}\mathbf{]} \qquad (4.8)$$

Therefore, the probability of the observed choice can be written as:

$$\Phi[2\gamma_j(c_j - w_i)] \qquad (4.10)$$

where $2\gamma_j(c_j - w_i) > 0$ if the legislator is on the same side of the cutting plane as the policy outcome she voted for (this is picked up by the signed distance term, γ_j).

For the quadratic utility model the probability of the observed choice can also be written in the more traditional form:

$$\Phi\left[\frac{2\gamma_j(c_j - w_i)}{\sigma_i}\right] \qquad (4.12)$$

where σ_i^2 is the legislator specific variance, that is:

$$\varepsilon_{ijn} - \varepsilon_{ijy} \sim N(0, \sigma_i^2)$$

This model is explored in depth in Poole (2001).

The likelihood function is:

$$L = \prod_{i=1}^{p}\prod_{j=1}^{q}\prod_{\tau=1}^{2} P_{ij\tau}^{C_{ij\tau}} \qquad (4.15)$$

where τ is the index for Yea and Nay, $P_{ij\tau}$ is the probability of voting for choice τ, and $C_{ij\tau} = 1$ if the legislator's actual choice is τ and zero otherwise.

It is standard practice to work with the natural log of the likelihood function:

$$\mathcal{L} = \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{\tau=1}^{2} C_{ij\tau} \ln P_{ij\tau} \qquad (4.16)$$

The Quadratic-Normal (QN) model is isomorphic with the standard IRT model. Specifically, using equations (4.5) to (4.7) let

$$\alpha_j = \left(\sum_{k=1}^{s} O_{jkn}^2 - \sum_{k=1}^{s} O_{jky}^2\right) = 2\gamma_j \mathbf{Z_j}'\mathbf{N_j} \quad \text{and}$$

$$\boldsymbol{\beta}_j = \begin{bmatrix} -2(O_{j1n} - O_{j1y}) \\ -2(O_{j2n} - O_{j2y}) \\ \vdots \\ -2(O_{jsn} - O_{jsy}) \end{bmatrix} = -2\gamma_j \mathbf{N_j} \qquad (4.30)$$

This transformation allows the difference between the latent utilities for Yea and Nay to be written in the same form as the item response model; namely:

$$y_{ij}^* = U_{ijy} - U_{ijn} = \alpha_j + \mathbf{X_i}'\boldsymbol{\beta}_j + \varepsilon_{ij} \qquad (4.31)$$

where y_{ij}^* is the difference between the latent utilities and

$$\varepsilon_{ij} = \varepsilon_{ijn} - \varepsilon_{ijy} \sim N(0, 1)$$

Let τ=Yea, then

$$\mathbf{P_{ij\tau} = \Phi[}u_{ijy} - u_{ijn}\mathbf{] =} \; \Phi\left[\frac{2\gamma_j(c_j - w_i)}{\sigma_i}\right] \mathbf{=} \; \Phi\left(\alpha_j + X_i'\beta_j\right)$$

To set up the Bayesian Framework let the prior distribution for the legislator ideal points be:

$$\mathbf{X_i} \sim N(\mathbf{0}, \mathbf{I_s}) = \boldsymbol{\xi}(\mathbf{X_i}) \qquad (4.34)$$

Where $\mathbf{0}$ is a s length vector of zeroes and $\mathbf{I_s}$ is an s by s identity matrix. The prior distribution for the roll call outcome parameters is:

$$\begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} \sim N(\mathbf{b_0}, \mathbf{B_0}) = \xi(\alpha_j, \boldsymbol{\beta_j}) \qquad (4.35)$$

where $\mathbf{b_0}$ is an s+1 length vector and $\mathbf{B_0}$ is a s+1 by s+1 variance-covariance matrix. Clinton, Jackman, and Rivers (2004) set $\mathbf{b_0}$ to a vector of zeroes and $\mathbf{B_0}$ to ηI_{s+1} where η is a large positive constant (typically 25). The posterior distribution is:

$$\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{X} \mid \mathbf{Y}) \propto \mathbf{f}(\mathbf{Y} \mid \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{X})\xi(\mathbf{X})\xi(\boldsymbol{\alpha}, \boldsymbol{\beta}) \propto \mathbf{L}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{X} \mid \mathbf{Y})\xi(\mathbf{X})\xi(\alpha, \boldsymbol{\beta}) \qquad (4.36)$$

The conditional distributions for $\boldsymbol{\xi}(\mathbf{Y}^*, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{X} \mid \mathbf{Y})$ that implement equation (4.36) are:

1) $$\mathbf{g}_{y_{ij}^*}\left(\mathbf{y}_{ij}^* \mid \boldsymbol{\alpha}_j, \boldsymbol{\beta}_j, \mathbf{X}_i, \mathbf{y}_{ij}\right) = \begin{cases} \mathbf{N}_{[0,\infty)}(\boldsymbol{\alpha}_j + \mathbf{X}_i'\boldsymbol{\beta}_j, 1) \text{ if } \mathbf{y}_{ij} = \mathbf{Yea} \\ \mathbf{N}_{(-\infty,0]}(\boldsymbol{\alpha}_j + \mathbf{X}_i'\boldsymbol{\beta}_j, 1) \text{ if } \mathbf{y}_{ij} = \mathbf{Nay} \\ \mathbf{N}_{(-\infty,\infty)}(\boldsymbol{\alpha}_j + \mathbf{X}_i'\boldsymbol{\beta}_j, 1) \text{ if } \mathbf{y}_{ij} = \mathbf{Missing} \end{cases}$$

where the subscript on the normal distribution indicates the range. For Yea and Nay the normal is truncated as indicated and the missing data is sampled over the entire real line.

2) $$\mathbf{g}_{\alpha,\beta}\left(\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j \mid \mathbf{Y}_j^*, \mathbf{X}, \mathbf{Y}_j\right) = \mathbf{N}(\boldsymbol{\nu}_j, \boldsymbol{\Xi}_j)$$

where Y_j^* and $\mathbf{Y_j}$ are the j[th] columns of $\mathbf{Y^*}$ and $\mathbf{Y}$, respectively, $\boldsymbol{\nu_j}$ is an s+1 length vector of means, and $\boldsymbol{\Xi_j}$ is the s+1 by s+1 variance-covariance matrix. Specifically,

$$\boldsymbol{\nu_j} = \boldsymbol{\Xi_j}\left[\mathbf{X}^{*'}\mathbf{Y}_j^* + \mathbf{B}_0^{-1}\mathbf{b}_0\right]$$

and

$$\boldsymbol{\Xi_j} = \left[\mathbf{X}^{*'}\mathbf{X}^* + \mathbf{B}_0^{-1}\right]^{-1}$$ where $\mathbf{X^*}$ is the p by s+1 matrix of legislator ideal points bordered by ones; that is, $\mathbf{X^* = [J_p|X]}$ and $\mathbf{J_p}$ is a p length vector of ones.

3) $\mathbf{g_X}\left(\mathbf{X_i} \mid \boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{Y_i^*},\mathbf{Y_i}\right) = \mathbf{N}(\mathbf{t_i},\mathbf{T_i})$

where Y_i^* and $\mathbf{Y_i}$ are the i[th] rows of $\mathbf{Y^*}$ and $\mathbf{Y}$, respectively, $\mathbf{t_i}$ is an s length vector of means, and $\mathbf{T_i}$ is the s by s variance-covariance matrix. Specifically,

$$\mathbf{t_i} = \mathbf{T_i}\left[\boldsymbol{\beta}'(\boldsymbol{\alpha} - \mathbf{Y_i^*})\right]$$

and

$$\mathbf{T_i} = \left[\boldsymbol{\beta}'\boldsymbol{\beta} + \mathbf{I_s}\right]^{-1}$$

Note that if a more general prior distribution is used for the legislator ideal points, namely:

$$\mathbf{X_i} \sim \mathrm{N}(\mathbf{t_0}, \mathbf{T_0})$$

Where $\mathbf{t_0}$ is a s length vector of means and $\mathbf{T_0}$ is a s by s variance-covariance matrix, then the expressions for $\mathbf{t_i}$ and $\mathbf{T_i}$ are:

$$\mathbf{t_i} = \mathbf{T_i}\left[\boldsymbol{\beta}'(\boldsymbol{\alpha} - \mathbf{Y_i^*}) + \mathbf{T_0^{-1}t_0}\right]$$

and

$$\mathbf{T_i} = \left[\boldsymbol{\beta}'\boldsymbol{\beta} + \mathbf{T_0}\right]^{-1}$$

At the h[th] iteration the MCMC sampling procedure outlined in equation set (4.29) for these conditional distributions is:

1) Sample $y_{ij}^{*(h)}$ from $\mathbf{g_{y_{ij}^*}}\left(\mathbf{y_{ij}^*} \mid \boldsymbol{\alpha}_j^{(h-1)},\boldsymbol{\beta}_j^{(h-1)},\mathbf{X}_i^{(h-1)},\mathbf{y_{ij}}\right)$, i=1,…,p , j=1,…,q

2) Sample $\boldsymbol{\alpha}_j^{(h)},\boldsymbol{\beta}_j^{(h)}$ from $\mathbf{g_{\alpha,\beta}}\left(\boldsymbol{\alpha_j},\boldsymbol{\beta_j} \mid \mathbf{Y}_j^{*(h-1)},\mathbf{X}^{(h-1)},\mathbf{Y_j}\right)$, j=1,…,q (4.37)

3) Sample $\mathbf{X}_i^{(h)}$ from $\mathbf{g_X}\left(\mathbf{X_i} \mid \boldsymbol{\alpha}^{(h-1)},\boldsymbol{\beta}^{(h-1)},\mathbf{Y}_i^{*(h-1)},\mathbf{Y_i}\right)$, i=1,…,p

Sampling from these conditional distributions is straightforward because they are all normal distributions with known means and variances.

This boils down to three simple OLS type steps:

1) Given $\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j, \mathbf{X}_i, \mathbf{y}_{ij}$, sample $\mathbf{y}^*_{ij}$ (the latent Yea/Nay variable) from the appropriate normal distribution:

$$\mathbf{g}_{y^*_{ij}}\left(\mathbf{y}^*_{ij} \mid \boldsymbol{\alpha}_j, \boldsymbol{\beta}_j, \mathbf{X}_i, \mathbf{y}_{ij}\right) = \begin{cases} \mathbf{N}_{[0,\infty)}(\boldsymbol{\alpha}_j + \mathbf{X}_i'\boldsymbol{\beta}_j, 1) \text{ if } \mathbf{y}_{ij} = \mathbf{Yea} \\ \mathbf{N}_{(-\infty,0]}(\boldsymbol{\alpha}_j + \mathbf{X}_i'\boldsymbol{\beta}_j, 1) \text{ if } \mathbf{y}_{ij} = \mathbf{Nay} \\ \mathbf{N}_{(-\infty,\infty)}(\boldsymbol{\alpha}_j + \mathbf{X}_i'\boldsymbol{\beta}_j, 1) \text{ if } \mathbf{y}_{ij} = \mathbf{Missing} \end{cases}$$

2) Given $\mathbf{X}_i, \mathbf{y}_{ij}, \mathbf{y}^*_{ij}$, sample $\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j$ from the normal distribution:

$$\mathbf{g}_{\alpha,\beta}\left(\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j \mid \mathbf{Y}^*_j, \mathbf{X}, \mathbf{Y}_j\right) = \mathbf{N}\left(\left[\mathbf{X}^{*'}\mathbf{X}^* + \mathbf{B}_0^{-1}\right]^{-1}\left[\mathbf{X}^{*'}\mathbf{Y}^*_j + \mathbf{B}_0^{-1}\mathbf{b}_0\right], \left[\mathbf{X}^{*'}\mathbf{X}^* + \mathbf{B}_0^{-1}\right]^{-1}\right)$$

3) Given $\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j, \mathbf{y}^*_{ij}, \mathbf{y}_{ij}$ sample $\mathbf{X}_i$ from the normal distribution:

$$\mathbf{g}_X\left(\mathbf{X}_i \mid \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{Y}^*_i, \mathbf{Y}_i\right) = \mathbf{N}\left(\left[\boldsymbol{\beta}'\boldsymbol{\beta} + \mathbf{I}_s\right]^{-1}\left[\boldsymbol{\beta}'(\boldsymbol{\alpha} - \mathbf{Y}^*_i)\right], \left[\boldsymbol{\beta}'\boldsymbol{\beta} + \mathbf{I}_s\right]^{-1}\right)$$